New Look at Wave Analogy for Prediction of Bubble Terminal Velocities

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The analogy between waves on the surface of an infinite fluid and bubbles rising in low-viscosity fluids of infinite extent, originally proposed by Mendelson for 3-D bubbles, has been used to predict the terminal velocity of plane bubbles. In terms of its terminal velocity, a plane bubble rising in a rectangular duct of small aspect (spacing-to-width) ratio behaves as if it were a 3-D bubble rising in an infinite medium as long as the end walls (the walls in the widthwise direction) are sufficiently far apart. As the end walls are moved toward each other, a wall effect is found to exist. A general expression for the terminal velocity of a bubble of any size rising in a rectangular duct including this wall effect is also developed based on the wave analogy and shown to compare well with existing data.

Introduction

The terminal velocity of individual bubbles in various media has been the subject of many experimental (Haberman and Morton, 1953; Peebles and Garber, 1953; Uno and Kintner, 1956; Collins, 1965; Maneri and Zuber, 1974) and analytical (Dumitrescu, 1943; Davies and Taylor, 1950; Garabedian, 1957; Birkhoff and Carter, 1957) investigations. In addition, a bubble rise model was developed by Mendelson (1967) for distorted (oblate spheroidal) and spherical cap bubbles rising in low viscosity fluids, based on the similarity he observed between the measured terminal velocities of these bubbles and the analytical form for the velocity of waves on the surface of a fluid. Maneri and Mendelson (1968) extended this analogy to include both three-dimensional (3-D) and plane (2-D) bubbles rising in infinite and finite media. Plane bubbles are formed between infinite parallel plates (infinite media) and in rectangular ducts (finite media) when the bubble volume is large enough that the bubble is in contact with both plates or the larger faces of the rectangular duct. In all cases, the bubble geometry investigated is compatible with the medium, namely, 3-D bubbles (spherical, spherical cap, cylindrical, and so on) rising in infinite media or circular tubes, and plane bubbles rising between infinite parallel plates or in rectangular ducts. To the author's knowledge, no studies have been performed for the more general case of a bubble of any size or geometry rising in a duct of rectangular cross section.

It is the intent of this article to build upon the wave analogy investigations of Mendelson, and Maneri and Mendelson to develop a rational methodology for determining the terminal velocity of these bubbles.

3-D Bubbles in Infinite Media

Mendelson was the first to recognize that a direct analogy exists between the propagation velocity of surface waves on a deep liquid and the terminal velocity of 3-D bubbles in infinite media. As will be shown, the equation for the wave velocity C presented by Mendelson is actually a special case of the more general expression (Kinsman, 1965) for the wave velocity at an interface between two fluids of different densities ρ_1 and ρ_2 and different depths h_1 and h_2:

$$C = \left[\frac{g\dfrac{\lambda}{2\pi}\Delta\rho + \dfrac{2\pi}{\lambda}\sigma}{\rho_1 \coth\dfrac{2\pi h_1}{\lambda} + \rho_2 \coth\dfrac{2\pi h_2}{\lambda}} \right]^{1/2} \tag{1}$$

Here λ is the wavelength and σ is the interfacial or surface tension.

For the case where the depth of both fluids is large relative to the wavelength (h_1/λ and $h_2/\lambda \gg 1$) and the lighter fluid is air ($\rho_1 = \rho_g \ll \rho_2 = \rho_l$) Eq. 1 becomes

$$C = C_\infty = \left[\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\lambda\rho_l}\right]^{1/2} \quad (2)$$

Mendelson converted this relationship into the terminal velocity U_∞ for bubbles rising in infinite media by replacing the wavelength with the perimeter p of the maximum cross section of an equivalent sphere whose volume V is equal to that of the bubble; thus,

$$\lambda = p = 2\pi r \quad (3)$$

where

$$r = \left[\frac{3V}{4\pi}\right]^{1/3} \quad (4)$$

The terminal velocity is therefore given by

$$U_\infty = \left[gr + \frac{\sigma}{r\rho_l}\right]^{1/2} \quad (5)$$

If, on the other hand, neither of the fluid densities can be disregarded, although h_1/λ and $h_2/\lambda \gg 1$ is still true, Eq. 1 reduces to

$$C_\infty = \left[\frac{g\lambda\Delta\rho}{2\pi(\rho_l + \rho_g)} + \frac{2\pi\sigma}{\lambda(\rho_l + \rho_g)}\right]^{1/2} \quad (6)$$

The corresponding terminal bubble velocity would be

$$U_\infty = C_{\infty(\lambda = 2\pi r)} = \left[gr\frac{\Delta\rho}{(\rho_l + \rho_g)} + \frac{\sigma}{r(\rho_l + \rho_g)}\right]^{1/2} \quad (7)$$

where $\Delta\rho$ is the density difference. When Eq. 7 is recast in the form:

$$(\rho_l + \rho_g)U_\infty^2 = gr\Delta\rho + \frac{\sigma}{r} \quad (8)$$

it becomes evident that it represents a balance between inertial, gravity and surface tension forces. As noted by Marrucci et al. (1970) in their application of the wave analogy to the motion of liquid drops in non-Newtonian systems, the dispersed phase (bubbles or drops) should be considered to be in steady motion; consequently, it does not contribute any inertia in contrast with the original classical wave problem. The density of the dispersed phase in the inertial term should therefore be eliminated, so that Eq. 7 becomes

$$U_\infty = \left[gr\frac{\Delta\rho}{\rho_l} + \frac{\sigma}{r\rho_l}\right]^{1/2} \quad (9)$$

Note that when $\rho_g \ll \rho_l$, Eq. 9 becomes the Mendelson form given by Eq. 5.

Although Eq. 9 compares well with terminal velocities measured in pure, single-component fluids, it tends to underpredict terminal velocities measured in multicomponent systems in the distorted bubble regime where surface tension effects predominate (Mendelson, 1967). Mendelson speculated that this deviation was associated with a dynamic vs. static surface tension effect. He noted that in pure, single-component fluids the dynamic surface tension and the static surface tension, as measured by the capillary rise method, appear to be the same, whereas in contaminated and purified multicomponent systems the two surface tensions may differ. In support of this speculation, he cited Jontz and Myers (1960) who reported large differences in the measured surface tension between dynamic and static interfaces.

The results of Mendelson led Fan and Tsuchiya (1990) to generalize Eq. 9 by replacing σ with $c\sigma$ where c is a constant that varies from 1.0 to 2.0 depending on the fluid. Thus, Eq. 9 becomes

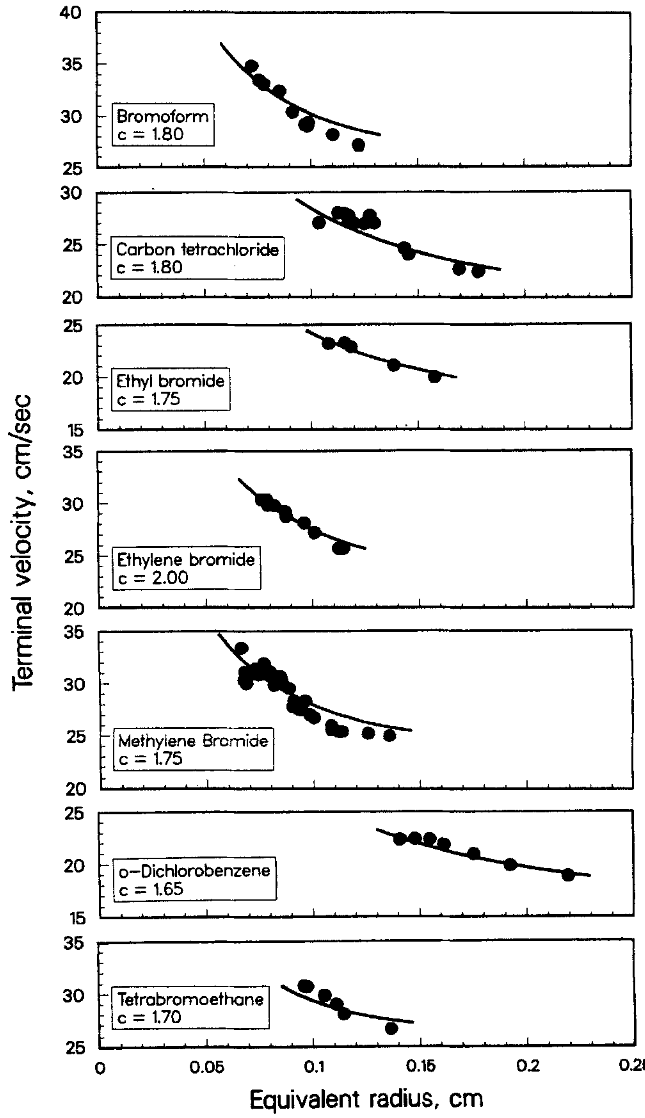

Figure 1. Comparison of wave analogy (Eq. 10) with terminal velocities of liquid drops in water obtained by Thorsen et al. (1968).

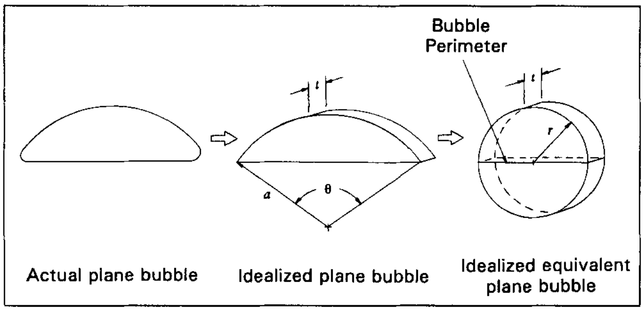

Figure 2. Definition of radius and perimeter of idealized equivalent plane bubble.

$$U_\infty = \left[gr\frac{\Delta\rho}{\rho_l} + \frac{c\sigma}{r\rho_l} \right]^{1/2} \tag{10}$$

With this generalization, the wave analogy not only compares well with the bubble terminal velocities in multicomponent fluids (Fan and Tsuchiya, 1990) but also with the terminal velocities of drops in fluids (Figure 1) when ρ_l and ρ_g are replaced by the densities of the continuous and dispersed phases respectively. In addition, Marrucci et al. (1970) showed that Eq. 9 ($c = 1.0$) compares well with the terminal velocity of chlorobenzene drops in water and non-Newtonian fluids. The comparisons of Figure 1 and the results of Marrucci et al. validate the elimination of ρ_g (the dispersed-phase density) from Eq. 7. Since the densities of both the continuous and dispersed phases are of the same order of magnitude in liquid-liquid systems, if Eq. 7 had been used, the drop terminal velocities would have been substantially underpredicted (up to $\sim 60\%$).

Plane Bubbles in Infinite Media

Maneri and Mendelson extended the wave analogy to plane bubbles rising between infinite parallel plates. This is the 2-D analog of the 3-D infinite media case previously considered. As illustrated in Figure 2, they idealized the plane bubble by transforming it into a disk having a thickness t equal to the plate separation and an equivalent radius given by

$$r = \left[\frac{V}{\pi t} \right]^{1/2} \tag{11}$$

For this case, the perimeter used to replace the wavelength is given by

$$p = 4r + 2t \tag{12}$$

so that the terminal velocity becomes

$$U_\infty = \left\{ g\left[\frac{2r+t}{\pi} \right] \frac{\Delta\rho}{\rho_l} + \left[\frac{\pi}{2r+t} \right] \frac{\sigma}{\rho_l} \right\}^{1/2} \tag{13}$$

The plane bubble terminal velocity data of Collins (1965) was used to qualify Eq. 13. Since Collins did not directly measure bubble volumes, Maneri and Mendelson used the geometrical relationship for the area of a 2-D circular segment and photographic data of Collins to estimate the 2-D equivalent radius (see Appendix). Within the uncertainties of this approach, Eq. 13 was shown to compare favorably with the Collins data. A more critical test of Eq. 13 is obtained relative to the 2-D data of Maneri (1970). He measured bubble terminal velocities in test sections having nominal widths of 6.45, 15.24 and 86.4 cm (see Table 1 for actual dimensions) and accurately determined the bubble volume either by direct displacement or by a geometric technique. The latter technique was used only with the 86.4-cm wide test section. It involved orienting the test section at 45 degrees after each run and determining the volume of the isosceles triangle formed by the air in the uppermost corner of the test section. The comparison given in Figure 3 shows that Eq. 13 tends to

Table 1. Test Section Sizes Used by Maneri

Cross-Sectional Dimensions (cm)	
Nominal	Actual
0.953×6.45	1.032×6.27
1.270×6.45	1.382×6.25
0.953×15.24	1.013×15.24
1.270×15.24	1.336×15.33
0.953×86.40	1.018×86.40
1.270×86.40	1.338×86.40

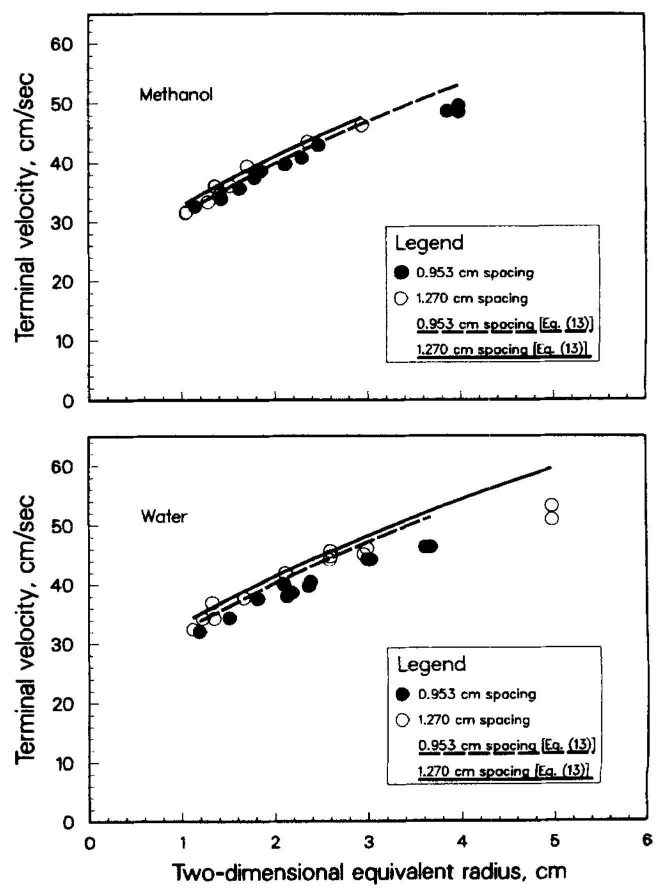

Figure 3. Comparison of Maneri-Mendelson expression with infinite media bubble terminal velocity data of Maneri (1970).

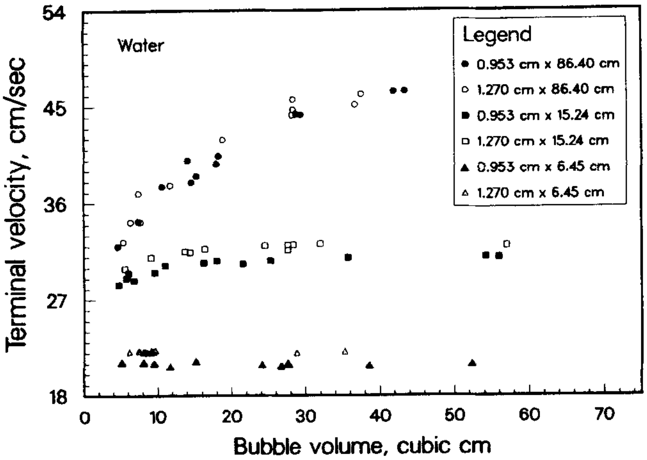

Figure 4. Observed effect of tank size on bubble terminal velocity (Maneri, 1970).

overpredict the data from the 86.4-cm wide test section (considered to be an infinite medium) with the deviation increasing with increasing bubble size. Furthermore, when the data of Maneri are plotted as a function of bubble volume (see Figure 4), a clear spacing effect exists for the 6.35- and 15.2-cm-wide ducts where the entire range of bubble sizes occupies a significant fraction of the duct cross-sectional area; however, the spacing effect disappears for the 86.4-cm-wide test section (infinite medium).

The latter observation suggests that for bubbles rising between infinite parallel plates (infinite 2-D media), the bubble volume governs the terminal bubble velocity independent of the spacing between the plates. In view of this, it is hypothesized that 2-D bubbles rising between infinite parallel plates behave as if they were 3-D bubbles rising in a fluid of infinite extent. To test this hypothesis, the Maneri infinite media data are compared in Figure 5 against the expression for 3-D bubbles (Eq. 9) with the bubble equivalent radius being determined from the measured bubble volume using Eq. 4. It is evident that, relative to the comparison given in Figure 3, a significant improvement is achieved in predicting the data.

With regard to the Collins data, no absolute experimental values for the terminal velocity in his "infinite" medium are given. He does give an experimental value in terms of an equation of $0.545\,[ga]^{1/2}$, where a is the frontal radius of curvature of the bubble, and states that the minimum bubble size in his 84-cm-wide test section is 3.81 cm. This gives a terminal velocity of 0.333 m/s. Collins also measured the included angle of the bubbles under the reasonable assumption that they are circular segments. Within the scatter of the angle data, it is estimated that a bubble in an infinite medium has an included angle of $\sim 108°$. With this information, the bubble volume corresponding to $a = 3.81$ cm can be estimated from which it is found that $r = 1.008$ cm (see Appendix). This value of r in Eq. 9 results in a predicted terminal velocity of 0.326 m/s which is within 2.3% of the experimental value of Collins. In view of the experimental scatter and the approximations required to convert Collins data to a spherical equivalent radius, this agreement is considered quite good.

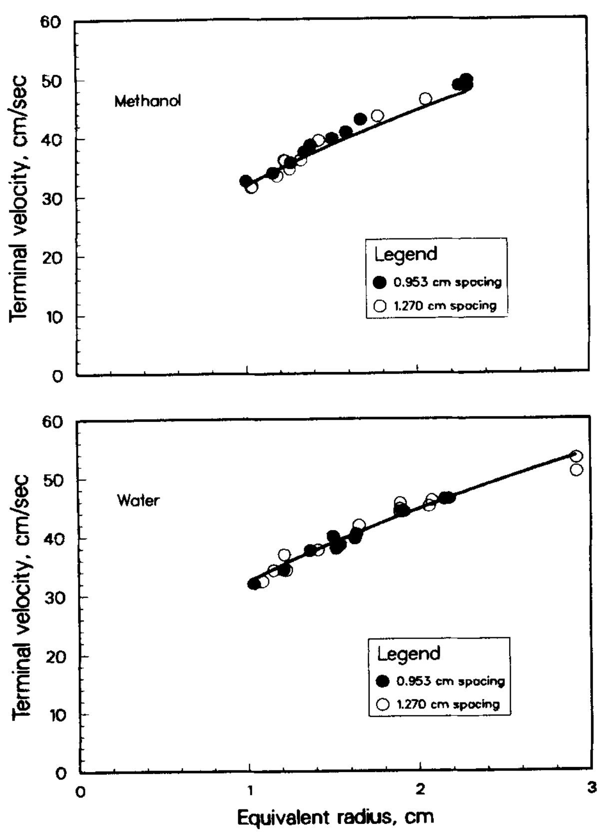

Figure 5. Comparison of wave analogy (Eq. 9) with infinite media bubble terminal velocities obtained by Maneri (1970).

Bubbles Rising in Rectangular Ducts

So far, it has been demonstrated that to within experimental accuracy, the terminal velocity of a bubble rising between two infinite parallel plates is unaffected by the presence of the plates regardless of the size of the bubble and its geometry (3-D or plane). On the other hand, the terminal velocity of bubbles rising in a rectangular duct (see Figure 6) of low

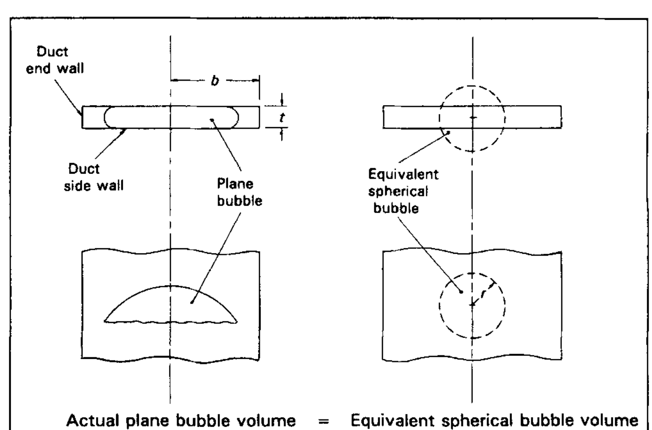

Figure 6. Definition of equivalent spherical bubble for rectangular duct geometry.

aspect ratio (spacing-to-width ratio < 0.5) will be affected by the proximity of the duct end walls (the walls defining the duct width). The effect of these walls on the terminal velocity is developed from the wave analogy as follows.

The expression for the wave velocity analogous to bubbles rising in finite media is obtained from Eq. 1 with $h_1/\lambda \gg 1$, $h_2 = h$, $\rho_2 = \rho_l$ and $\rho_1 = \rho_g$; thus,

$$C = \left[\frac{g\dfrac{\lambda}{2\pi}\Delta\rho + \dfrac{2\pi}{\lambda}\sigma}{\rho_g + \rho_l \coth \dfrac{2\pi}{\lambda}h} \right]^{1/2} \tag{14}$$

The bubble terminal velocity is therefore

$$U = C_{(\lambda = p)} = \left[\left(g\frac{p}{2\pi}\frac{\Delta\rho}{\rho_l} + \frac{2\pi}{p}\frac{\sigma}{\rho_l} \right) \tanh \frac{2\pi h}{p} \right]^{1/2} \tag{15}$$

where the same argument used for the infinite media case has been invoked to eliminate ρ_g from the denominator of Eq. 14 (the inertial term).

Based on the results for infinite media, p is given by Eq. 3 and h is assumed to be proportional to the channel half-width b:

$$h = c_1 b \tag{16}$$

This assumption is in keeping with the earlier observation that the terminal velocity is unaffected by the duct side walls (the walls defining the duct spacing). With the use of Eqs. 3 and 16 the terminal velocity is given by

$$U = \left[\left(gr\frac{\Delta\rho}{\rho_l} + \frac{\sigma}{\rho_l r} \right) \tanh \frac{c_1 b}{r} \right]^{1/2} = U_\infty \left[\tanh \frac{c_1 b}{r} \right]^{1/2} \tag{17}$$

The constant c_1 is determined from the condition that when r reaches a certain fraction of the duct half-width, say βb, the slug terminal velocity limit U_s is reached, or

$$U_{(r=\beta b)} = U_s = U_{\infty(r=\beta b)} \left[\tanh \frac{c_1}{\beta} \right]^{1/2} \tag{18}$$

This yields for the constant

$$c_1 = \beta \tanh^{-1} \left[\frac{U_s}{U_{\infty(r=\beta b)}} \right]^2 \tag{19}$$

An expression for the slug rise velocity is given by

$$U_s = 0.3458[1 + 0.41(t/b) - 0.092(t/b)^2] \left[gb \frac{\Delta\rho}{\rho_l} \right]^{1/2} \tag{20}$$

which was obtained from a fit of the data of Griffith (1963) and Maneri (1970). The velocity ratio in Eq. 19 is obtained from Eq. 20 and Eq. 9 evaluated at $r = \beta b$ or

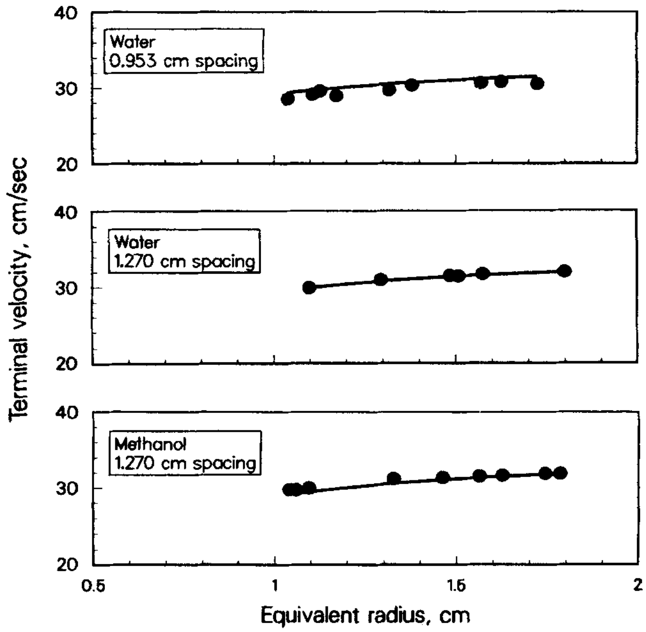

Figure 7. Comparison of wave analogy (Eq. 17) with bubble terminal velocities obtained by Maneri (1970) in a 15.24-cm-wide tank.

$$\frac{U_s}{U_{\infty(r=\beta b)}} = 0.3458 \frac{[1 + 0.41(t/b) - 0.092(t/b)^2]}{\left[\beta + \dfrac{\sigma}{\beta \Delta\rho g b^2} \right]^{1/2}} \tag{21}$$

The last remaining parameter β is determined from the experimental results of Maneri (1970). From the data obtained on the 15.24-cm-wide duct, it is estimated that the slug terminal velocity limit is reached when the bubble volume, measured directly by a displacement method, is ~ 25 cc which corresponds to a bubble equivalent radius of ~ 1.8 cm. This results in a value of β of 0.235.

The terminal velocity for bubbles rising in rectangular ducts can now be determined from Eq. 17 in conjunction with Eqs. 19 and 21 for $r \leq 0.235b$ or from Eq. 20 for $r > 0.235b$. At the present time, no rectangular duct data are available for bubbles whose diameter is less than or equal to the duct spacing. Maneri (1970) measured terminal velocity data for bubble volumes ≥ 5 cm^3 ($d \geq 2.0$ cm) in his 15.24-cm-wide duct for nominal spacings of 0.953 and 1.27 cm. These data, obtained for both water and methanol, are compared with prediction in Figure 7 where it is seen that good agreement is obtained.

To the author's knowledge, Collins (1965) is the only other investigator who has measured plane bubble terminal velocities in finite media; however, he gives only one absolute value of the terminal velocity corresponding to a specific set of conditions: $U = 0.36$ m/s for $a = 6.1$ cm, $b = 12.7$ cm and $\theta = 110°$. This geometric information results in a 3-D equivalent radius of 1.403 cm. The predicted value for this case from Eq. 17 is 0.348 m/s which compares well (within 3.5%) with the experimental value.

It should be emphasized that Eq. 17 is applicable to bubbles rising in low viscosity fluids in rectangular ducts of low aspect ratio (< 0.5) with the spacing being within the range where liquid flows around the bubble along both the end walls and side walls of the duct. As the spacing is reduced, a critical value t_c is reached where the liquid no longer flows along the side walls. At this point, a significant change occurs in the terminal velocity which is not predicted by Eq. 17. This cessation of side wall flow was first reported by Griffith (1963). He conducted slug rise experiments in rectangular ducts at atmospheric conditions and observed that flow along the side walls did not occur for a duct having a spacing of 0.33 cm and a width of 4.52 cm. Griffith attributed this behavior to the predominance of capillary forces across the spacing as the spacing is reduced toward zero.

Maneri (1970) converted this observation into an empirical criterion which states that flow in a low aspect ratio duct will cease along the side walls when

$$t < t_c = 1.2167\left[\frac{\sigma}{\Delta\rho g}\right]^{1/2} \tag{22}$$

This criterion, modeled after an analytical result originally derived by Bretherton (1961) for tubes, defines the range of applicability of Eq. 17 with regard to duct spacing.

Conclusions

It has been shown that the wave analogy as originally developed by Mendelson for bubbles rising in low viscosity fluids in infinite 3-D media is directly applicable to infinite 2-D media. In other words, the terminal velocity of bubbles rising in infinite media is independent of the bubble geometry (plane vs. 3-D). This means that the bubble perimeter $2\pi r$ based on an equivalent spherical bubble (the key element of the wave analogy) need not be redefined in terms of a 2-D geometry, as was done by Maneri and Mendelson (1968). Furthermore, the same holds true for bubbles rising in rectangular ducts of low aspect ratio (< 0.5) so long as the duct spacing is greater than the critical spacing given by Eq. 22. Finally, it has been demonstrated that the wave analogy for infinite media as modified by Fan and Tsuchiuya for bubbles rising in multicomponent systems can also be used to adequately predict droplet terminal velocities.

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Notation

a = frontal radius of curvature of bubble
A = area of circular segment
b = half-width of rectangular duct
c = Fan-Tsuchiya surface tension coefficient
c_1 = proportionality constant in Eq. 16
C = wave velocity at an interface of two fluids
d = bubble equivalent diameter
g = gravitational acceleration
h = fluid depth in wave equation
p = perimeter of equivalent bubble
r = bubble equivalent radius
t = plate separation or duct spacing
t_c = critical duct spacing for zero liquid flow along duct side walls

Greek letters

β = ratio of bubble equivalent radius to duct half-width
θ = included angle of a circular segment
ρ = fluid density

Subscripts

g = gas
l = liquid
s = slug
∞ = infinite media

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Appendix

The area A of a circular segment is given by the relation:

$$A = \frac{a^2(\theta - \sin\theta)}{2} \tag{A1}$$

where a is the radius of the circle and θ is the included angle of the segment. The radius of a circle having the same area is then

$$r = \left[\frac{A}{\pi}\right]^{1/2} \tag{A2}$$

or

$$r = a\left[\frac{\theta - \sin\theta}{2\pi}\right]^{1/2} \tag{A3}$$

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